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Module Bases for Multivariate Splines

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We characterize module bases of spline spaces in terms of their determinants, degree sequences, and dimension series. These characterization also provide tests for freeness of the module. Applications are given to the basis and dimension problem for spline spaces. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let Δ be a polyhedral subdivision of a region in Euclidean space. A C^r-spline on Δ is a piecewise polynomial function on Δ which is continuously differentiable up to order r. We denote the set of all C^r splines of degree at most k by $C^r_k(\Delta)$. A problem in approximation theory is the computation of bases and dimensions of the vector spaces $C^r_k(\Delta)$ (cf. [1, 2, 4]).

We approach this problem by studying the set $C^r(\Delta)$ consisting of all splines of arbitrary degree. $C^r(\Delta)$ has a natural $S = \mathbf{R}[x_1, ..., x_d]$ module structure via pointwise multiplication. By studying the algebraic properties of $C^r(\Delta)$, we have been able to glean information about the $C^r_k(\Delta)$'s, in many cases simultaneously for all k. In [5], we showed that a reduced module basis for $C^r(\Delta)$ will give rise to bases of the spline spaces $C^r_k(\Delta)$. In [5, 8] we have gave combinatorial and topological conditions for $C^r(\Delta)$ to be free and for a reduced basis to exists. In [3], we showed that the generating function of the dimension sequence of the spline spaces is the Hilbert series of the graded version or homogenisation of $C^r(\Delta)$. When Δ is a simplicial subdivision, we have shown that the homogenization of $C^0(\Delta)$ is the face ring of a simplicial complex, providing a purely combinatorial interpretation of $C^0(\Delta)$ [5].

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In this work, we characterize module bases in terms of their determinants and in the graded or reduced case, in terms of the degrees of the basis elements. If the module contains a set of elements satisfying these conditions, then it will be free. We also show that the dimension series of $C^r(\Delta)$ must have a certain form in order for a reduced basis to exist. See also [6] for a study of free and reduced bases for spline spaces.

1.1. *Preliminaries*

Let Δ be a polyhedral subdivision of a region in $\mathbf{R}^d$. Δ can be described as a union of d-dimensional convex polytopes such that the intersection of any two polytopes is a face of each. We will call Δ a **d-complex**. If v is a vertex of Δ, we define the **star of v**, denoted $\mathrm{st}_\Delta(v)$, to be the union of d-polytopes containing v together with their faces. We say Δ is a **central** complex if it is the star of one of its vertices. The **join** of a d-polytope P with a vertex v outside the affine span of P is the convex hull of P and v in $\mathbf{R}^{d+1}$. We will denote this by $\hat{P}$. Similarly, $\hat{\Delta}$ will denote the join of the d-complex Δ with a vertex in $\mathbf{R}^{d+1}$. Note that $\hat{\Delta}$ is always a central complex.

DEFINITION 1.1. A complex is *strongly connected* if every two d-polytopes are connected by a path that goes through faces of codimension 1. If Δ is connected and $\mathrm{st}_\Delta(v)$ is strongly connected for all v, then we say Δ is *hereditary*.

We now give a formal definition of $C^r(\Delta)$. Let $S = \mathbf{R}[x_1, ..., x_d]$.

DEFINITION 1.2. For a positive integer r and a d-complex Δ, $C^r(\Delta)$ is the set of r-differentiable functions $F\colon \Delta \to \mathbf{R}$ such that for every d-polytope σ, F restricted to σ is given by a polynomial in S.

Although $C^r(\Delta)$ is an infinite dimensional vector space over $\mathbf{R}$, each $C^r_k(\Delta)$ is finite dimensional. The *dimension series* of $C^r(\Delta)$ is the generating function of the sequence $\{\dim_{\mathbf{R}} C^r_k(\Delta)\}$, i.e.,

$$\mathscr{D}(r, \Delta) = \sum_{k=0}^{\infty} \dim_{\mathbf{R}} C^r_k(\Delta)\, t^k.$$

Notation. Let Δ_i denote the set of i-dimensional faces of Δ, and let Δ_i^0 be the set of i-dimensional interior faces of Δ. Similarly, $f_i(\Delta) = \#\Delta_i$, and $f_i^0(\Delta) = \#\Delta_i^0$.

By Theorem 3.1 of [5], if $C^r(\Delta)$ is free for any r, then Δ must be hereditary. For this reason, we restrict our study to hereditary complexes. In this case, there is an easy way to characterize elements of $C^r(\Delta)$. Choose an ordering $\sigma_1, ..., \sigma_n$ of Δ_d. With respect to this ordering, a spline function can be represented as an n-tuple of polynomials, $(f_1, ..., f_n)$, where f_i is the restriction of F to the face σ_i. If $\tau = \sigma_i \cap \sigma_j$ has dimension $d-1$, then the ideal of polynomials which vanish on τ is generated by an affine form, denoted l_τ. Note that l_τ is unique up to constant multiple. Another way to think of this is as follows: The affine span of τ, aff(τ), is a hyperplane in $\mathbf{R}^d$ and l_τ is an affine form whose kernel is that hyperplane. The following proposition is proved in [3] in a more general case.

PROPOSITION 1.3. *If Δ is hereditary and $F = (f_1, ..., f_n)$ is a piecewise polynomial function on Δ, then F is in $C^r(\Delta)$ if and only if whenever $\tau = \sigma_i \cap \sigma_j$ has dimension $d-1$, l_τ^{r+1} divides $f_i - f_j$.*

DEFINITION 1.4. Let M be an S module consisting of n-tuples of polynomials. M is *free* if it has a basis over S. A basis $\mathcal{B} = \{B_1, ..., B_n\}$ is called *reduced* if every element F of M can be written in the form $\sum_{i=1}^n s_i B_i$, where for each e, $\deg(s_i B_i) \leqslant \deg(F)$.

It follows that if $\mathcal{B}$ is a reduced basis for $C^r(\Delta)$, then for each k, the set

$$\mathcal{B}_k = \{mB_i \colon m \text{ is a monomial in } S, \deg(mB_i) \leqslant k\}$$

will be a vector space basis for $C^r_k(\Delta)$ ([5, Proposition 6.2]). We showed in [5] that a reduced basis exists if and only if $C^r(\hat{\Delta})$ is free. When Δ is central, $C^r(\Delta)$ will be graded [5] and any homogeneous basis will be reduced. In fact, there is a strong connection between reduced and homogeneous bases. Let $F \in S^n$ and let F^h denote the homogenization of F in $S[z]^n$, where z is a new variable. For example, if $F = (x^2y - 2x, y+1)$, then $F^h = (x^2y - 2xz^2, yz^2 - z^3)$. Similarly, if $G \in S[z]^n$, then $G(1) = G|_{z=1} \in S^n$.

PROPOSITION 1.5. *Let $\mathcal{B} \subset C^r(\Delta)$. Then $\mathcal{B}$ is a reduced basis for $C^r(\Delta)$ if and only if $\mathcal{B}^h$ is a homogeneous basis for $C^r(\hat{\Delta})$.*

Proof. This statement follows from the proof of Theorem 6.3 of [5], and the fact that $\mathcal{B}^h(1) = \mathcal{B}$ and if $\mathcal{H}$ is a basis for $C^r(\hat{\Delta})$, then $\mathcal{H}(1)^h = \mathcal{H}$. ∎

2. DETERMINANTS, DEGREES, AND BASES

In this section we characterize the determinant and the degree sequence of a module basis for $C^r(\Delta)$. We can use these characterizations to compute a basis for $C^r(\Delta)$ and simultaneously for all $C^r_k(\Delta)$ in the reduced case. In particular, this gives a method for determining whether $C^r(\Delta)$ is free. See [5, 8] for characterizations of freeness in terms of the combinatorics and topology of Δ. Since we are only concerned with the free case, we assume Δ is hereditary.

The idea for these characterizations comes from Saito [9] who proved similar results in the case of divisors. See also [10].

Let $S = \mathbf{R}[x_1 \cdots x_d]$ and let Δ be a hereditary d-complex. Recall from Section 1 that elements of $C^r(\Delta)$ may be viewed as f_d-tuples of S, given an ordering of Δ_d. In [5] we showed that $C^r(\Delta)$ has rank f_d, which means that in the free case, any basis will contain exactly f_d elements. Thus, the concept of the determinant of a basis is well defined.

Lemma 2.1. *Let τ and τ' be distinct elements of Δ^0_{d-1}. If $\mathrm{aff}(\tau) = \mathrm{aff}(\tau')$, then τ and τ' cannot both be faces of the same d-face σ.*

Proof. If τ and τ' are faces of σ, then $\tau = \mathrm{aff}(\tau) \cap \sigma$ and $\tau' = \mathrm{aff}(\tau') \cap \sigma$. Since τ and τ' are distinct, $\mathrm{aff}(\tau)$ cannot be the same hyperplane as $\mathrm{aff}(\tau')$. ∎

Let Q denote the product of $\{(l_\tau)^{r+1}\}$, where τ ranges over Δ^0_{d-1}.

Proposition 2.2. *Let $\{F_1, ..., F_n\} \in C^r(\Delta)$. Then $Q = \Pi(l_\tau)^{r+1}$ divides* $\det[F_1, ..., F_n]$.

Proof. Let $\tau \in \Delta^0_{d-1}$, say $\tau = \sigma_1 \cap \sigma_2$, and let $F_i = (f_{1i}, ..., f_{ni})^{\mathrm{T}}$. Then

$$\det[F_1, ..., F_n] = \begin{vmatrix} f_{1i} & \cdots & f_{1n} \\ f_{2i} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix} = \begin{vmatrix} f_{1i} - f_{2i} & \cdots & f_{1n} - f_{2n} \\ f_{2i} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix}.$$

For each i, $(l_\tau)^{r+1}$ divides $f_{1i} - f_{2i}$ (by Proposition 1.1), so l_τ^{r+1} divides $\det[F_1, ..., F_n]$. This is true for all τ. If the l_τ's are distinct, then they are pairwise relatively prime, so Q must divide $\det[F_1, ..., F_n]$. If the l_τ's are not distinct, we can do the following. Suppose $l_\tau = l_{\tau'}$. If $\tau = \sigma_1 \cap \sigma_2$ and $\tau' = \sigma_i \cap \sigma_j$, we must have i and j both greater than 2, by Lemma 2.1. Assume without loss of generality that $(i, j) = (3, 4)$. Then

$$\det[F_1, ..., F_n] = \begin{vmatrix} f_{1i} - f_{2i} & \cdots & f_{1n} - f_{2n} \\ f_{2i} & \cdots & f_{2n} \\ f_{2i} - f_{4i} & \cdots & f_{3n-4n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{vmatrix}$$

For each i, l_τ^{r+1} divides $f_{3i} - f_{4i}$, so $(l_\tau^{r+1})^2$ divides $\det[F_1, ..., F_n]$, etc. ∎

THEOREM 2.3. *$\{F_1, ..., F_n\}$ in $C^r(\Delta)$ form a basis over S if and only if* $\det[F_1, ..., F_n] = cQ$, *for some nonzero real constant c.*

Proof. Suppose $\det[F_1, ..., F_n] = Q$. Then $\{F_1, ..., F_n\}$ must be linearly independent over S. By Cramer's rule, $QS^n \subset (F_1, ..., F_n)$, the S-module generated by $F_1, ..., F_n$. Let $F \in C^r(\Delta) - \{0\}$. Then $QF \subset (F_1, ..., F_n)$, so $QF = \sum_{i=1}^n s_i F_i$ for some $\{s_i\}$ in S. Then

$$\begin{aligned} s_i Q &= s_i(\det[F_1, ..., F_n]) \\ &= \det[F_1 \cdots F_{i-1} s_i F_i F_{i+1} \cdots F_n] \\ &= \det[F_1 \cdots F_{i-1} \textstyle\sum s_j F_j F_{i+1} \cdots F_n] \\ &= \det[F_1 \cdots F_{i-1} QF F_{i+1} \cdots F_n] \\ &= Q(\det[F_1 \cdots F_{i-1} F F_{i+1} \cdots F_n]) \end{aligned}$$

which lies in $Q^2 S$ by Proposition 2.2. So Q divides s_i. Then $F = \sum (s_i/Q) F_i \in (F_1, ..., F_n)$.

Conversely, suppose $\{F_1, ..., F_n\}$ form a basis for $C^r(\Delta)$. By Proposition 2.2, $\det[F_1, ..., F_n] = sQ$, for some $s \in S - \{0\}$. Fix τ in Δ^0_{d-1}. Let $Q_\tau = Q/l_\tau^{r+1}$. If $\tau = \sigma_1 \cap \sigma_2$, then $(Q_\tau, Q_\tau, 0, ..., 0)$ is in $C^r(\Delta)$, as long as the l_τ's are distinct. Then

$$QQ_\tau^{n-1} = \begin{vmatrix} Q_\tau & & & & \\ Q_\tau & Q & & & \\ 0 & 0 & Q_\tau & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & Q_\tau \end{vmatrix}.$$

This determinant is equal to rsQ, for some r in $S - \{0\}$, since each column is in $C^r(\Delta)$, and so can be written as a combination of the F_i's. Thus s divides Q_τ^n. If the l_τ's are distinct, then s must be a constant, since τ was arbitrary. Suppose the l_τ's are not distinct, for example, $l_\tau = l_{\tau'}$. By Lemma 2.1, we may assume that $\tau = \sigma_1 \cap \sigma_2$ and $\tau' = \sigma_3 \cap \sigma_4$. Let $\tilde{Q}_\tau = Q_\tau + l_\tau^{r+1}$. Then

$$Q\tilde{Q}_\tau^{n-1} = \begin{vmatrix} \tilde{Q}_\tau & & & & \\ \tilde{Q}_\tau & Q_\tau & & & \\ 0 & 0 & \tilde{Q}_\tau & & \\ 0 & 0 & \tilde{Q}_\tau & Q_\tau & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \cdots & \tilde{Q}_\tau \end{vmatrix}.$$

Thus s divides $\tilde{Q}_\tau^{n-1}$. Since τ was arbitrary, s must be a constant. ∎

When Δ is central, we can always find a homogeneous basis (and hence a reduced basis) for $C^r(\Delta)$. In this case, a basis is determined by the degrees of the basis elements.

COROLLARY 2.4. *Let Δ be a central complex. A set of linearly independent homogeneous elements $\{F_1, ..., F_n\}$ in $C^r(\Delta)$ form a basis over R if and only if $\sum_{i=1}^n \deg(F_i) = \deg Q$.*

Proof. Since the F_i's are homogeneous, the degree of $\det[F_1, ..., F_n]$ is either 0 or the sum of the degrees of the F_i's. Since they are independent, the degree cannot be 0. By Theorem 2.3, the F_i's form a basis if and only if $\deg(\det[F_1, ..., F_n]) = \deg(Q)$. ∎

COROLLARY 2.5. *Let Δ be a d-complex. A set of linearly independent elements $\{F_1, ..., F_n\}$ in $C^r(\Delta)$ form a reduced basis over R if and only if $\sum_{i=1}^n \deg(F_i) = \deg Q$.*

Proof. By Theorem 6.3 of [5], if $C^r(\Delta)$ has a reduced basis, $\mathscr{B}$, then the homogenization $\mathscr{B}^h$ will be a homogeneous basis for $C^r(\hat{\Delta})$. By Corollary 2.4 we obtain the right-hand side. Conversely, given the right-hand side, if we homogenize each F_i, then the new set will be a homogeneous basis for $C^r(\hat{\Delta})$, again by Corollary 2.4. By Theorem 6.3 of [5], the dehomogenized elements will be the original F_i's and they will form a reduced basis. ∎

In particular, in order to find a reduced basis, it suffices to find a set of f_d linearly independent splines whose degrees sum to the degree of $Q = f_{d-1}^0(r+1)$. This often turns out to be easier than finding a reduced generating set and trimming it to a basis.

EXAMPLE 1. Let Δ be the octahedron in $\mathbf{R}^3$ with vertices at the unit vectors $\pm e_i$, triangulated by putting a vertex at the origin. $C^0(\Delta)$ is free by Theorem 4.5 of [5]. In fact, $C^r(\Delta)$ is free for all r, and a basis is given as follows: Let X_i denote the unique piecewise linear function on Δ that $X_i(v_j) = \delta_{ij}$, the Kronecker delta. A basis for $C^r(\Delta)$ is given by

$$\{1, X_3^{r+1}, (X_3X_4)^{r+1}, X_5^{r+1}, (X_4X_5)^{r+1}, (X_3X_5)^{r+1}, (X_3X_4X_5)^{r+1}\}.$$

Since Δ is central, we see that the sum of the degrees $= 12(r+1) = f_{d-1}^0(r+1)$.

EXAMPLE 2. Let Δ be an annulus in the plane subdivided into three quadrilaterals with interior edges defined by the linear forms x, y, and $y - x - 1$. $C^r(\Delta)$ is free for all r, by Theorem 3.5 of [5], but $C^r(\hat{\Delta})$ is never free, by Theorem 4.2 of [8]. Thus, a basis for $C^r(\Delta)$ will never be reduced. Using the computer algebra system COCOA, we compute a basis for $C^0(\Delta)$ to be

$$B_1 = (1, 1, 1)$$

$$B_2 = (0, -xy, y(y-x-1))$$

$$B_3 = (0, -x(x-1), x(y-x-1)).$$

The determinant of $(B_1, B_2, B_3) = xy(y-x-1) = Q$, so by Theorem 2.3, this is a basis for $C^0(\Delta)$. However, the sum of the degrees of the B_i's is 4, and $\deg(Q) = 3$, showing that this basis is not reduced.

3. Reduced Bases and Dimension Series

We can often determine whether $C^r(\Delta)$ has a reduced basis from its dimension series $\mathcal{D}(\Delta, r)$. Recall that in the central case, having a reduced basis is equivalent to freeness.

PROPOSITION 3.1 [3]. *$\mathcal{D}(\Delta, r)$ has the form*

$$\frac{P(C^r(\Delta), t)}{(1-t)^{d+1}},$$

where $P(C^r(\Delta), t)$ is a polynomial in t with integer coefficients.

THEOREM 3.2. *Let Δ be a d-complex. Suppose $C^r(\Delta)$ is free with reduced basis $\{B_1, B_2, ..., B_n\}$. If $P(C^r(\Delta), t) = \sum a_i t^i$, then $a_i = \#\{B_k : \deg(B_k) = i\}$. In particular, if $C^r(\Delta)$ has a reduced basis, then the coefficients of $P(C^r(\Delta), t)$ are nonnegative.*

Proof. Let $\mathcal{B} = \{B_1, ..., B_t\}$ be a reduced basis of $C^r(\Delta)$. Then every element of F of $C^r(\Delta)$ can be written in the form $\sum_{i=1}^{n} s_i B_i$, where $\deg(s_i B_i) \leqslant \deg(F)$. Thus, a basis for $C^r_k(\Delta)$ will consist of

$$\{mB : m \text{ is a monomial}, B \in \mathcal{B}, \text{and } \deg(mB) \leqslant k \leqslant \deg(F)\}.$$

The result now follows from counting these basis elements. ∎

EXAMPLE 3. Let Δ be the octahedron with vertices $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, -1)$, and $(1, 1, 1)$. $C^0(\Delta)$ is free by Theorem 4.5

of [5] and has a reduced basis because Δ is central. Using the computer algebra system MACAULAY, we compute

$$P(C^1(\Delta), t) = 1 + t^2 + 4t^3 + t^5 + 2t^6 - t^7.$$

This has a negative coefficient so, by Theorem 3.2, $C^1(\Delta)$ is not free.

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